Chapter 13 Vectors

1. In this question all distances are in km.

   A ship P sails from a point A, which has position vector $\begin{pmatrix}0\\0\end{pmatrix}$, with a speed of 52 kmh^{-1} in the direction of $\begin{pmatrix}-5\\12\end{pmatrix}$.

   (a) Find the velocity vector of the ship. [1]

   $\sqrt{25+144} = 13$

   velocity vector $= \begin{pmatrix}-20\\48\end{pmatrix}$

   (b) Write down the position vector of P at a time t hours after leaving A. [1]

   $t\begin{pmatrix}-20\\48\end{pmatrix}$

   At the same time that ship P sails from A, a ship Q sails from a point B, which has position vector $\begin{pmatrix}12\\8\end{pmatrix}$, with velocity vector $\begin{pmatrix}-25\\45\end{pmatrix}$

   (c) Write down the position vector of Q at a time t hours after leaving B. [1]

   $\begin{pmatrix}12\\8\end{pmatrix} + t\begin{pmatrix}-25\\45\end{pmatrix}$

   (d) Using your answers to **parts (b)** and **(c)**, find the displacement vector $\overline{PQ}$ at time t hours. [1]

   $\vec{PQ} = \vec{OQ} - \vec{OP}$

   $= \begin{pmatrix}12-25t\\8+45t\end{pmatrix} - \begin{pmatrix}-20t\\48t\end{pmatrix}$

   $= \begin{pmatrix}12-5t\\8-3t\end{pmatrix}$

(e) Hence show that $PQ = \sqrt{34t^2 - 168t + 208}$.

[2]

$$\sqrt{(12-5t)^2 + (8-3t)^2}$$

$$= \sqrt{144 - 120t + 25t^2 + 64 - 48t + 9t^2} = \sqrt{34t^2 - 168t + 208} \text{ (shown)}$$

(f) Find the value of t when P and Q are first 2 km apart.

[2]

$$34t^2 - 168t + 208 = 4$$

$$34t^2 - 168t + 204 = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{168 \pm \sqrt{(168)^2 - 4(34)(204)}}{2(34)}$$

$= 2.79$ or 2.15

(reject)

2. The position vectors of three points, A, B and C, relative to an origin O, are $\overbrace{\begin{pmatrix} -5 \\ -7 \end{pmatrix}, \begin{pmatrix} 10 \\ -4 \end{pmatrix}}^{*}$ and $\begin{pmatrix} x \\ y \end{pmatrix}$ respectively. Given that $\overrightarrow{AC} = 4\overrightarrow{BC}$, find the unit vector in the direction of $\overrightarrow{OC}$.

[5]

$$\overrightarrow{AC} = 4\overrightarrow{BC}$$

$$\overrightarrow{OC} - \overrightarrow{OA} = 4(\overrightarrow{OC} - \overrightarrow{OB})$$

$$\overrightarrow{OC} - \overrightarrow{OA} = 4\overrightarrow{OC} - 4\overrightarrow{OB}$$

$$4\overrightarrow{OB} - \overrightarrow{OA} = 4\overrightarrow{OC} - \overrightarrow{OC}$$

$$4\overrightarrow{OB} - \overrightarrow{OA} = 3\overrightarrow{OC}$$

$$4\begin{pmatrix} 10 \\ -4 \end{pmatrix} - \begin{pmatrix} -5 \\ -7 \end{pmatrix} = 3\overrightarrow{OC}$$

$$\begin{pmatrix} 40 \\ -16 \end{pmatrix} - \begin{pmatrix} -5 \\ -7 \end{pmatrix} = 3\overrightarrow{OC}$$

$$\begin{pmatrix} 45 \\ -9 \end{pmatrix} = 3\overrightarrow{OC}$$

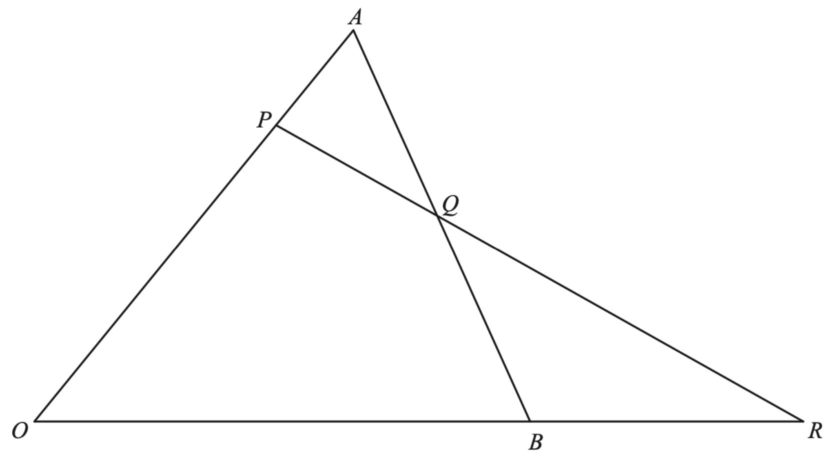
$$\overrightarrow{OC} = \begin{pmatrix} 15 \\ -3 \end{pmatrix}$$

$$|\overrightarrow{OC}| = 3\sqrt{26}$$

$$\text{unit vector} = \frac{1}{3\sqrt{26}}\begin{pmatrix} 15 \\ -3 \end{pmatrix}$$

$$= \frac{1}{\sqrt{26}}\begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

3.



The diagram shows a triangle *OAB* such that $\vec{OA}$ = **a** and $\vec{OB}$ = **b**. The point *P* lies on *OA* such that $\vec{OP} = \frac{3}{4}\vec{OA}$. The point Q is the mid-point of *AB*. The lines *OB* and *PQ* are extended to meet at the point *R*. Find, in terms of **a** and **b**,

a. $\vec{AB}$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= b - a$

[1]

b. $\vec{PQ}$, Give your answer in its simplest form.

$\vec{PQ} = \vec{PO} + \vec{OB} + \vec{BQ}$

$= -\frac{3}{4}a + b - \frac{1}{2}b + \frac{1}{2}a$

$= -\frac{1}{4}a + \frac{1}{2}b$

[3]

It is given that $n\vec{PQ} = \vec{QR}$ and $\vec{BR} = k\mathbf{b}$, where n and k are positive constants.

c. Find $\vec{QR}$ in terms of n, **a** and **b**.

$\vec{QR} = n\vec{PQ} = n\left(-\frac{1}{4}a + \frac{1}{2}b\right)$

[1]

d. Find $\vec{QR}$ in terms of k, **a** and **b**.

$\vec{QR} = \vec{QB} + \vec{BR}$

[2]

$= \frac{1}{2}b - \frac{1}{2}a + kb$

e. Hence find the value of n and of k.

$-\frac{1}{4}na + \frac{1}{2}nb = -\frac{1}{2}a + \left(\frac{1}{2} + k\right)b$

[3]

$-\frac{1}{4}n = -\frac{1}{2}$

$n = 2$

$\frac{1}{2} + k = \frac{1}{2} \times 2$

$\frac{1}{2} + k = 1$

$k = \frac{1}{2}$

4.

(a) Find the unit vector in the direction of $\begin{pmatrix} 5 \\ -12 \end{pmatrix}$.

$$\frac{1}{13}\begin{pmatrix} 5 \\ -12 \end{pmatrix}$$

[1]

(b) Given that $\begin{pmatrix} 4 \\ 1 \end{pmatrix} + k\begin{pmatrix} -2 \\ 3 \end{pmatrix} = r\begin{pmatrix} -10 \\ 5 \end{pmatrix}$, find the value of each of the constants *k* and *r*.

[3]

$4-2k = -10r$

$1+3k = 5r \quad \times 2$

$2+6k = 10\cancel{r}$

$4-2k = -\cancel{10}r$

$6+4k = 0$

$k = -\frac{3}{2}$

$5r = 1-\frac{9}{2}$

$r = -\frac{7}{10}$

(c) Relative to an origin *O*, the points *A*, *B* and *C* have position vectors **p**, 3**q**-**p** and 9**q**-5**p** respectively.

(i) Find $\overrightarrow{AB}$ in terms of **p** and **q**.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 3q-p-p = 3q-2p$

[1]

(ii) Find $\overrightarrow{AC}$ in terms of **p** and **q**

[1]

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= 9q-5p-p$

$= 9q-6p$

(iii) Explain why *A*, *B* and *C* all lie in a straight line.

[1]

$3\overrightarrow{AB} = \overrightarrow{AC}$

$\therefore$ A, B, C are collinear.

(iv) Find the ratio *AB* : *BC*.

[1]

$BC = 9q - 5p - 3q + p$

$= 6q - 4p$

$= 2(3q - 2p)$

$AB : BC = 1 : 2$

5. The vectors **a** and **b** are such that **a** = $\underset{=}{a}$**i**+**j** and **b** = 12**i**+$\underset{=}{b}$**j**.

α β

(a) Find the value of each of the constants a and b such that 4**a**-**b** = (α +3)**i**-2**j**. [3]

$$4(\alpha i + j) - 12i - \beta j = (\alpha+3)i - 2j$$

$$(4\alpha - 12)i + (4-\beta)j = (\alpha+3)i - 2j$$

$4\alpha - 12 = \alpha + 3$ $\quad$ $4 - \beta = -2$

$3\alpha = 15$ $\quad$ $-\beta = -6$

$\alpha = 5$ $\quad$ $\beta = 6$

(b) Hence find the unit vector in the direction of **b** - 4**a**. [2]

$$b - 4a = 12i + 6j - 4(5i + j)$$

$$= 12i + 6j - 20i - 4j$$

$$= -8i + 2j$$

$$\text{unit vector} = \frac{1}{2\sqrt{17}} \times \begin{pmatrix} -8 \\ 2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{17}} \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

6. A particle *P* is initially at the point with position vector $\begin{pmatrix}30\\10\end{pmatrix}$ and moves with a constant speed of 10 ms^{-1} in the same direction as $\begin{pmatrix}-4\\3\end{pmatrix}$.

a. Find the position vector of *P* after *t* s.

$\sqrt{16+9} = 5$

velocity vector $= \begin{pmatrix}-8\\6\end{pmatrix}$

$P = \begin{pmatrix}30\\10\end{pmatrix} + \begin{pmatrix}-8\\6\end{pmatrix}t$

[3]

As *P* starts moving, a particle *Q* starts to move such that its position vector after *t* s is given by $\begin{pmatrix}-80\\90\end{pmatrix} + t\begin{pmatrix}5\\12\end{pmatrix}$.

b. Write down the speed of *Q*.

$\sqrt{25+144} = 13$

[1]

c. Find the exact distance between *P* and *Q* when *t* = 10, giving your answer in its simplest surd form.

$P = \begin{pmatrix}30\\10\end{pmatrix} + \begin{pmatrix}-8\\6\end{pmatrix}10 = \begin{pmatrix}-50\\70\end{pmatrix}$

$Q = \begin{pmatrix}-80\\90\end{pmatrix} + \begin{pmatrix}5\\12\end{pmatrix}10 = \begin{pmatrix}-30\\210\end{pmatrix}$

$\overrightarrow{PQ} = \begin{pmatrix}-30\\210\end{pmatrix} - \begin{pmatrix}-50\\70\end{pmatrix}$

$= \begin{pmatrix}20\\140\end{pmatrix}$

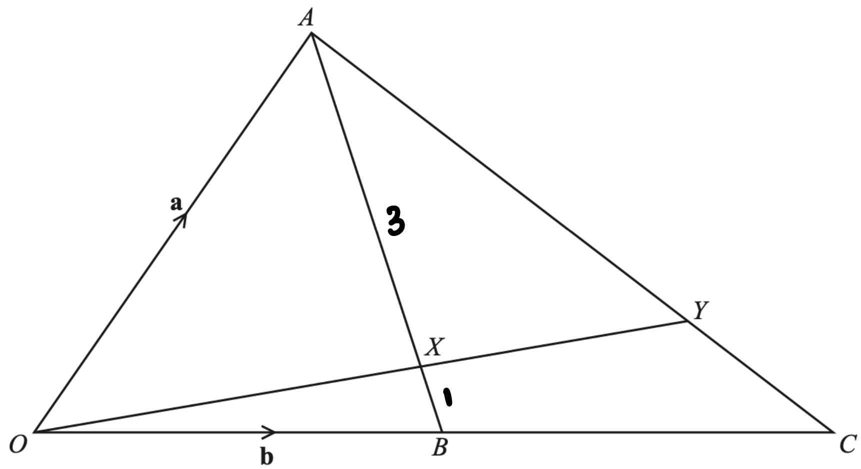
distance $= \sqrt{400+19600}$

$= \sqrt{20000}$

$= 100\sqrt{2}$

[3]

7.



The diagram shows the triangle OAC. The point B is the midpoint of OC. The point Y lies on AC such that OY intersects AB at the point X where $AX : XB$ = 3:1. It is given that $\vec{OA}$ = **a** and $\vec{OB}$ = **b**.

(a) Find $\vec{OX}$ in terms of **a** and **b**, giving your answer in its simplest form.

[3]

$\vec{AB} = -a + b$

$\vec{AX} = -\frac{3}{4}a + \frac{3}{4}b$

$\vec{OX} = a - \frac{3}{4}a + \frac{3}{4}b$

$= \frac{1}{4}a + \frac{3}{4}b$

(b) Find $\vec{AC}$ in terms of **a** and **b**.

$\vec{AC} = \vec{OC} - \vec{OA} = 2b - a$

[1]

(c) Given that $\vec{OY} = h\vec{OX}$, find $\vec{AY}$ in terms of **a**, **b** and **h**.

$\vec{AY} = \vec{AO} + \vec{OY}$ [1]

$= -a + h\vec{OX}$

$= -a + h(\frac{1}{4}a + \frac{3}{4}b)$

(d) Given that $\vec{AY} = m\vec{AC}$, find the value of *h* and of *m*.

$-a + \frac{1}{4}ha + \frac{3}{4}hb = m(2b - a)$ [4]

$-1 + \frac{1}{4}h = -m$

$\frac{3}{4}h = 2m$

$h = \frac{8}{3}m$

$-1 + \frac{1}{\cancel{4}} \times \frac{\cancel{8}^{2}}{3}m = -m$

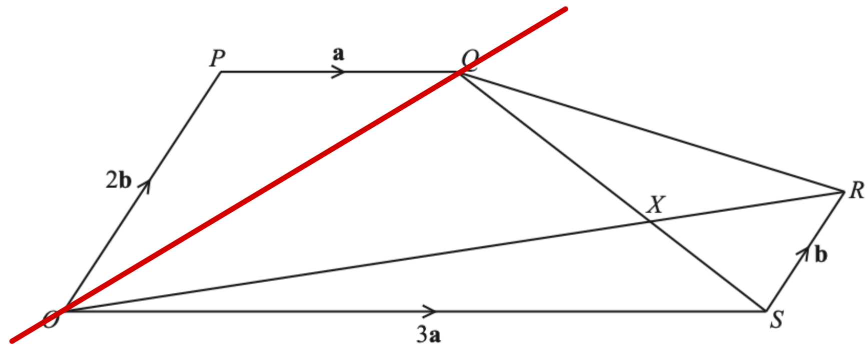
$\frac{2}{3}m - 1 = -m$

$\frac{5}{3}m = 1$

$m = \frac{3}{5}$

$h = \frac{8}{3} \times \frac{3}{5} = \frac{8}{5}$

8.



In the diagram $\vec{OP}$= 2**b**, $\vec{OS}$ = 3**a**, $\vec{SR}$= **b** and $\vec{PQ}$ = **a.** The lines *OR* and *QS* intersect at *X*.

(a) Find $\vec{OQ}$ in terms of **a** and **b**.

$\vec{OQ} = 2b + a$

[1]

(b) Find $\vec{QS}$ in terms of **a** and **b**.

$\vec{QS} = -2b - a + 3a$

$= 2a - 2b$

[1]

(c) Given that $\vec{QX} = \mu\vec{QS}$, find $\vec{OX}$ in terms of **a**, **b** and μ.

$\vec{OX} = 2b + a + \mu(2a - 2b)$

[1]

(d) Given that $\vec{OX} = \lambda\vec{OR}$, find $\vec{OX}$in terms of **a**, **b** and λ.

$\vec{OX} = \lambda(3a + b)$

[1]

(e) Find the value of μ and λ.

$\lambda(3a + b) = 2b + a + \mu(2a - 2b)$

[3]

$3\lambda = 1 + 2\mu$ $\quad\quad$ $\lambda = 2 - 2\mu$

$\lambda + 2\mu = 2$ — ②

$3\lambda - 2\cancel{\mu} = 1$ — ①

$\lambda + 2\cancel{\mu} = 2$

$4\lambda = 3$

$\lambda = \frac{3}{4}$

$2\mu = 2 - \frac{3}{4}$

$2\mu = \frac{5}{4}$

$\mu = \frac{5}{8}$

(f) Find the value of $\frac{QX}{XS}$.

[1]

$QX = \mu QS$

$QX = \frac{5}{8} QS$

$QX = 5$ unit

$XS = 3$ unit

$\frac{QX}{XS} = \frac{5}{3}$

(g) Find the value of $\frac{OR}{OX}$.

[1]

$OX = \lambda OR$

$OX = \frac{3}{4} OR$

$\frac{OR}{OX} = \frac{4}{3}$